Math 206A Lecture 1 Notes

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1 Overview of Course Topics

1.1 The Borsuk conjecture

Theorem 1.1 (Borsuk conjecture). *Let $P \subseteq \mathbb{R}^d$ be a compact, convex body. Then $P = \bigcup_{i=1}^{d+1} P_i$ such that* $\operatorname{diam}(P_i) < \operatorname{diam}(P)$.

Why does this make sense?

Theorem 1.2 (Borsuk). *This is true for $d = 2$. Moreover, there exists some $\varepsilon > 0$ such that* $\operatorname{diam}(P_i) < (1 - \varepsilon) \operatorname{diam}(P)$.

Try this out with a hexagon, and split it in three parts to get some intuition. We will actually prove this. However, we will not prove the following:

Theorem 1.3. *The Borsuk conjecture is true for $d = 3$.*

The Borsul conjecture is actually not always true. We will prove the following.

Theorem 1.4 (Kahn-Kalai,1990). *The Borsuk conjecture is false for $d > 2200$.*

The proof uses linear algebra methods in extremal combinatorics.

1.2 Convex polytopes

Let $P \subseteq \mathbb{R}^d$ be a convex polytope, and let $f_i(P)$ be the number of i-dimensional faces. What can be said about $(f_0, f_1, f_2, \dots, f_{d-1})$?

Example 1.1. For $d = 2$, a pentagon has vector $(5, 5)$.

Example 1.2. For $d = 3$, if we have 5 vertices, what vectors can we have? We can have $(5, 9, 6)$ (for a slice of cake shape) and $(5, 8, 5)$ (for a square pyramid shape).

In dimensions $d \geq 4$, we do not have a full picture of what is going on.

Theorem 1.5 (conjecture)**.** *There does not exist $P \subseteq \mathbb{R}^4$ with f-vector $(n, 10n, 10n, n)$.*

Definition 1.1. P is **simplicial** if every face is a simplex.

Theorem 1.6 (D-S)**.** *There exist $\lfloor n/2 \rfloor$ linear relations on f-vectors of simplicial polytopes in $\mathbb{R}^n$.*

Later, we will prove an inequality relating f_2, f_1, and f_0.

1.3 Rigidity

Here is a question. Let E be the edges of an icosahedron, and suppose $f : E \to \mathbb{R}_+$ such that $|f(e) - 1| < 1/100$. Does there exist a "perturbed icosahedron" with edge lengths $\{f(e)\}$? The answer is yes, due to a theorem of Dehn[1] from about 1912. In fact, this is true for every simplicial polytope.

1.4 Combinatorial geometry of curves

Let Q be an equilateral convex polygon (all sides have the same unit length).

Example 1.3. For quadrilaterals, we can have a rhombus or a square.

Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be angles of Q. We know that $\sum \alpha_i = (n-2)\pi$.

Theorem 1.7 (4 vertex theorem for polygonal curves)**.** *There exist at least 4 sign changes in $(\alpha_{i+1} - \alpha_i)$.*

We will see a geometric proof of this, and we will provide a combinatorial proof for the result in 3 dimensions. It actually gets simpler!

This actually implies the following theorem about smooth curves:

Theorem 1.8. *The curvature of a smooth, closed curve changes sign at least 4 times.*

[1] Dehn was a student of Hilbert.